Lecture II

27.

- → Need identify characteristic $\begin{cases} \text{collective} \\ \text{collisional} \end{cases}$ scales

- → plasma is continuum ⇒ characterize by collective modes (can calculate response).

II Plasma/Fluid Collective Modes, Response

$\begin{cases} \text{Why} \\ \epsilon = \text{?} \end{cases}$ as mode covered in Lect I.

a) Cold Plasma $(T, P \Rightarrow 0)$

$$\frac{\partial n}{\partial t} + \underline{\nabla} \cdot (n \underline{v}) = 0 \qquad \rightarrow \text{continuity}$$

$$n m \frac{d\underline{v}}{dt} = n q \left(\underline{E} + \frac{\underline{v}}{c} \times \underline{B} \right) \qquad \rightarrow \text{momentum balance}$$

+ Maxwell Equations

For electromagnetic/electrostatic wave:

$$\Rightarrow n = n_0 + \tilde{n}$$
$$\underline{v} = \cancel{\underline{v}_0} + \tilde{\underline{v}}$$
$$\underline{E} = \cancel{\underline{E}_0} + \tilde{\underline{E}}$$
$$B = \cancel{B_0} + \tilde{\underline{B}}$$

⇒ 1 species — ions stationary

$$\frac{\partial \tilde{n}}{\partial t} = -n_0 \, \underline{\nabla} \cdot \underline{\tilde{V}}$$

$$\frac{\partial \underline{\tilde{V}}}{\partial t} = \frac{q}{m} \underline{\tilde{E}} + \cancel{\frac{q \, \underline{\tilde{V}} \times \underline{\tilde{B}}}{mc}}$$

$$\underline{\nabla} \cdot \underline{\tilde{E}} = 4\pi q \tilde{n} \qquad \underline{\nabla} \cdot \underline{\tilde{B}} = 0$$

$$\underline{\nabla} \times \underline{\tilde{B}} = \frac{4\pi}{c} \underline{\tilde{J}} + \frac{1}{c} \frac{\partial \underline{\tilde{E}}}{\partial t} \qquad \underline{\nabla} \times \underline{\tilde{E}} = -\frac{1}{c} \frac{\partial \underline{\tilde{B}}}{\partial t}$$

$$\underline{\tilde{J}} = n_0 q \underline{\tilde{V}}$$

⇒ Fourier Transforming $\quad \underset{\text{etc}}{\underline{E}} = \sum_{\underline{k},\omega} \underline{E}_{\underline{k},\omega} e^{i(\underline{k}\cdot\underline{x} - \omega t)}$

$$\underline{k} \times \underline{\tilde{B}}_{\underline{k},\omega} = -\frac{4\pi n_0 q i}{c} \underline{\tilde{V}}_{\underline{k},\omega} - \frac{\omega}{c} \underline{\tilde{E}}_{\underline{k},\omega}$$

$$\frac{\underline{k}}{\omega} \times (\underline{k} \times \underline{E}_{\underline{k},\omega}) = \frac{4\pi n_0 q^2}{m_e \omega} \underline{E}_{\underline{k},\omega} - \frac{\omega}{c} \underline{E}_{\underline{k},\omega}$$

$$\underline{k}(\underline{k} \cdot \underline{E}_{\underline{k},\omega}) - k^2 \underline{E}_{\underline{k},\omega} = \underbrace{\underbrace{\frac{4\pi n_0 q^2}{c^2 m_e}}}_{\omega_p^2/c^2} \underline{E}_{\underline{k},\omega} - \frac{\omega^2}{c^2} \underline{E}_{\underline{k},\omega}$$

→ EM waves $(\underline{k} \cdot \underline{E}_{k\omega} = 0)$

$$k^2 \underline{E}_{k\omega} - \underline{k}(\underline{k} \cdot \underline{E}_{k\omega}) = \frac{\omega^2}{c^2} \underline{E}_{k\omega} - \frac{\omega_p^2}{c^2} \underline{E}_{k\omega}$$

$\omega_p^2 = 4\pi n_0 q^2 / m$ → plasma frequency
→ characteristic frequency for (non-neutralized) plasma oscillations
→ (ions stationary ⇒ $\omega \gg \omega_{pi}$ $(\sim 1/M_i)$)

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

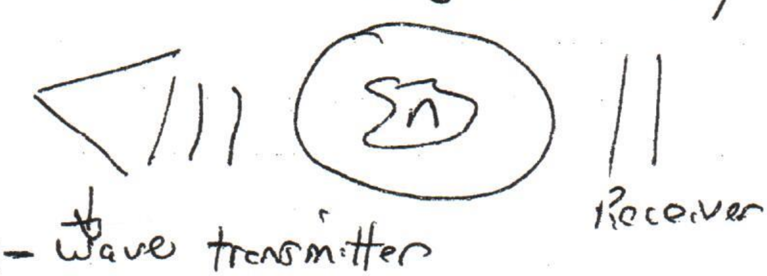
$$\omega^2 = \omega_p^2 + c^2 k^2$$

Dispersion Relation for EM Waves in Unmagnetized Plasma

→ $\epsilon(\omega) = 1 - \omega_p^2/\omega^2$ — cold plasma dielectric (dispersive)

→ $\omega < \omega_p \Rightarrow k^2 < 0$ — ω_p is cut-off frequency

→ can diagnose density



i) Electrostatic Waves / Oscillations (Langmuir Osc.)

$\underline{k}\cdot\underline{E} = k\underline{E}$ $\leftrightarrow$ alternatively obtain via $\left\{\begin{array}{l}\text{Fluid eqns} \\ + \\ \text{Gauss Law}\end{array}\right.$

$\Rightarrow \quad 0 = \left[(\omega^2-\omega_p^2)/c^2\right] \underline{E}_{k,0}$

$\omega^2 = \omega_p^2$

- ions stationary $\leftrightarrow$ $\omega^2 \gg \omega_{pi}^2 \sim 1/M_i$
- non-propagating oscillation $\omega^2=\omega_p^2$

b.) Warm Plasma Waves (Electrostatic) (Langmuir Waves)

Now, introduce pressure

$$m \frac{\partial \tilde{\underline{V}}}{\partial t} = q\,\tilde{\underline{E}} - \frac{\underline{\nabla}\,\tilde{p}}{n_0}$$

$$\frac{\partial \tilde{n}}{\partial t} = -n_0\, \underline{\nabla}\cdot\tilde{\underline{V}}$$

$$\underline{\nabla}\cdot\tilde{\underline{E}} = 4\pi q\,\tilde{n}$$

$\left\{\begin{array}{l} p = p_0 (n/n_0)^\gamma \\ \quad\text{- adiabatic} \\ p = \tilde{n}\, T \\ \quad\text{- isothermal}\end{array}\right.$

$\Downarrow$

determines eqn. state from kin. Th.

$$\frac{\partial^2 \tilde{n}}{\partial t^2} = -n_0 \left(\frac{q}{m} \nabla \cdot \underline{E} - \frac{\nabla^2 \tilde{p}}{n_0 m} \right)$$

$$= -\omega_p^2 \tilde{n} + \frac{T}{m} \nabla^2 \tilde{n}$$

$$\Rightarrow \frac{\partial^2 \tilde{n}}{\partial t^2} = \underbrace{-\omega_p^2 \tilde{n}}_{\text{plasma oscillation}} + \underbrace{\frac{T}{m} \nabla^2 \tilde{n}}_{\text{streaming induced by } \nabla p \text{ (a la' acoustics)}} \qquad \frac{T}{m} \equiv V_{the}^2$$

$$\omega^2 = \omega_p^2 + k^2 V_{Th}^2$$

$$\Rightarrow \omega^2 = \omega_p^2 (1 + k^2 \lambda_D^2)$$

λ_D → Debye Length

$$\lambda_D^2 \equiv V_{Th}^2 / \omega_p^2$$

i.e.

$$\nabla^2 \tilde{n} - \frac{1}{\lambda_D^2} \tilde{n} = \frac{1}{V_T^2} \frac{\partial^2 \tilde{n}}{\partial t^2} \Rightarrow \omega \to 0$$

recovers screened Gauss' Law

Recall Debye Length:

$$\nabla^2 \phi = \underbrace{4\pi \rho_{ind}}_{\text{plasma response (screening)}} + \underbrace{4\pi q\, \delta(\underline{x} - \underline{x}_0)}_{\text{particle response}}$$

↳ ~~test~~ particle ~~response~~ response.

$$f = n_0 \exp\left[-\frac{mv^2}{T} \pm \frac{q\phi}{T}\right]$$

$$\rho_{ind} = n_0 q \exp[q\phi/T_e] - n_0 q \exp[-q\phi/T_i]$$

$$\cong \frac{\omega_{pe}^2}{4\pi V_{Te}^2}\phi + \frac{\omega_{pi}^2}{4\pi V_{Ti}^2}\phi$$

$$\Rightarrow \nabla^2\phi - \left(\frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2}\right)\phi = 4\pi q\,\delta(\underline{x} - \underline{x}_0)$$

Then $\omega \ll \omega_p$ $\Rightarrow$ plasma response is streaming to screen test charge

$\Rightarrow$ hence appearance Debye length

$\omega \gg \omega_p$ $\Rightarrow$ warm plasma oscillation (too fast to screen)

Note: cold plasma $(T=0)$ $\Rightarrow$ no energy to move to screen charge

$\Rightarrow$ Warm Plasma Wave combines $\begin{cases} \text{plasma oscillation} \\ \oplus \text{ acoustic wave} \end{cases}$

electron.

i.e. carries wave momentum.

b.) Ion Acoustic Wave

- so far, 'single species' dynamics. Now 2 species

- consider now, ion acoustic wave, with
$V_{Ti} < \frac{\omega}{k} < V_{Te}$

Recall, for warm electrons:

$$\frac{\partial \tilde{n}}{\partial t} + \nabla\cdot(\cancel{\tilde{n}\tilde{v}}) = -n_0 \nabla\cdot\tilde{\underline{v}}$$

$$m_e \frac{\partial \tilde{v}}{\partial t} = -|e|\tilde{E} - T_e \frac{\nabla\tilde{n}}{n_0} \qquad (\tilde{p} = \tilde{n}\,T_e)$$

$$= +|e|\nabla\tilde{\phi} - T_e\frac{\nabla\tilde{n}}{n_0}$$

$\Rightarrow$

$$\frac{\partial^2\tilde{n}}{\partial t^2} = -n_0\left(\frac{|e|}{m_e}\nabla^2\tilde{\phi} - \frac{V_{Te}^2}{n_0}\nabla^2\tilde{n}\right)$$

$$\frac{\partial^2\tilde{n}}{\partial t^2} - V_{Te}^2\nabla^2\tilde{n} = -n_0\frac{|e|}{m_e}\nabla^2\tilde{\phi}$$

$\downarrow$ $O(\omega^2)$

$\downarrow$ electron compression $O(k^2 V_{Te}^2)$

– for: for $k^2 v_{Te}^2 \gg \omega^2$

$$\frac{\tilde{n}}{n_0} = \frac{|e|\hat{\phi}}{T_e}$$

Note:

→ equivalent to limit where electron inertia negligible

i.e. $m_e \to 0$ $\quad (v_{Te}^2 \to \infty) \Rightarrow \frac{\tilde{n}}{n} = \frac{|e|\hat{\phi}}{T_e}$

→ could, in limit $k^2 v_{Te}^2 \gg \omega^2$, obtain from Boltzmann response

i.e. $E \to E - |e|\hat{\phi} \Rightarrow f_e = c\exp\left[-\frac{\left(\frac{mv^2}{2} - |e|\hat{\phi}\right)}{T}\right]$

$$\cong \left(1 + \frac{|e|\hat{\phi}}{T}\right) f_{0M}$$

For ions (cold):

$$\frac{\partial \tilde{n}}{\partial t} = -n_0 \, \underline{\nabla} \cdot \tilde{\underline{v}}$$

$$\frac{\partial \tilde{\underline{v}}_i}{\partial t} = +\frac{|e|}{m_i} \tilde{\underline{E}}$$

$$\frac{\partial^2 \tilde{n}}{\partial t^2} = + n_0 \frac{|e|}{m_i} \nabla^2 \tilde{\phi}$$

$$\frac{\tilde{n}_{i\,k,\omega}}{n_0} = + \frac{|e|}{m_i} \frac{k^2}{\omega^2} \tilde{\phi}_{k,\omega}$$

$$\nabla^2 \phi = -4\pi n_0 |e| \left(\frac{\hat{n}_i}{n_0} - \frac{\hat{n}_e}{n_0} \right)$$

$$k^2 \hat{\phi}_{k,\omega} = +4\pi n_0 |e| \left(\frac{|e|}{m_i} \frac{k^2}{\omega^2} \hat{\phi}_{k,\omega} - \frac{|e|}{T_e} \hat{\phi}_{k,\omega} \right)$$

$$k^2 = \frac{\omega_{pi}^2}{\omega^2} k^2 - \underbrace{\frac{\omega_{pe}^2}{v_{Te}^2}}_{\to \lambda_{De}}$$

$$\Rightarrow \left(k^2 + 1/\lambda_{De}^2 \right) = \frac{\omega_{pi}^2}{\omega^2} k^2$$

$$\therefore \quad \boxed{\begin{aligned} \omega^2 &= k^2 c_s^2 / (1 + k^2 \lambda_{De}^2) \\ c_s^2 &= T_e / m_i \end{aligned}}$$

Note:

→ Compare hydrodynamic acoustic wave:

$$\frac{\partial \tilde{\rho}}{\partial t} = -\rho_0 \underline{\nabla} \cdot \underline{\tilde{V}} \quad ; \quad \frac{\partial \tilde{V}}{\partial t} = -\frac{\nabla \tilde{p}}{\rho_0}$$

$$\tilde{p} = c_s^2 \tilde{\rho}$$

	Hydro	Ion-Acoustic
"Springiness"	Gas Pressure	T_e
Inertia	Gas Density/inertia	m_i

i.e. ion-acoustic wave is two component, hybrid oscillation

$$\rightarrow \left(k^2 + 1/\lambda_{De}^2\right) = \frac{\omega_{pi}^2}{\omega^2} k^2$$

$$\left(1 + 1/k^2\lambda_{De}^2\right) = \frac{\omega_{pi}^2}{\omega^2}$$

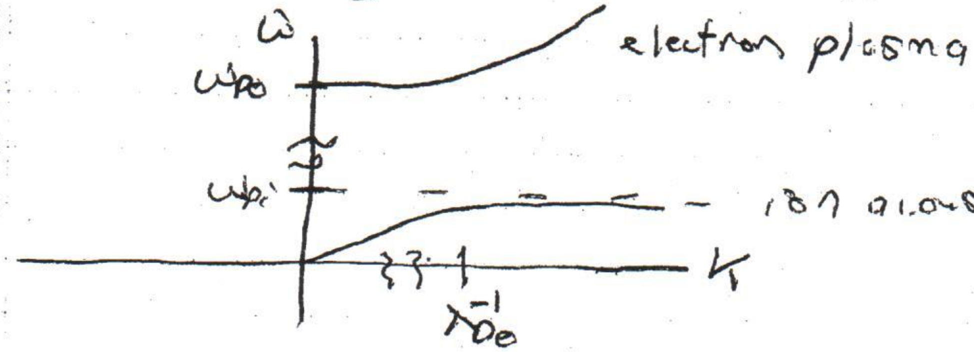
Debye shielding (T_e) ⟶ ion plasma oscillation (m_i)

Key pt.

$\boxed{\text{Ion-acoustic waves as Debye-shielded ion plasma oscillation}}$

Note : $k^2 \lambda_{De}^2 \geq 1 \Rightarrow \omega^2 \to \omega_{pi}^2$

$\Rightarrow$



Basic modes (electrostatic) of un-magnetized plasma.

Basic Scales : $\left\{ \begin{array}{l} \omega_{pe}, \ \omega_{pi} \\ \lambda_{De} \\ V_{Te}, \ C_s \end{array} \right.$

C.) Nonlinear Fluid Plasma Waves

$\Rightarrow$ Langmuir, Ion Acoustic Waves $\Rightarrow$ 1D Compressional Waves

$\Rightarrow$ 1D Compressional Wave (Linear)

$\downarrow$ (steepening – finite amplitude)

Shock

collisional (standard) (Burgers) — collisionless (KdV)